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Solutions of fractional multi-order integral and differential equations using a Poisson-type transform

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Abstract

We consider a wide class of integral and ordinary differential equations of fractional multi-orders $(1/\rho_1, 1/\rho_2, \ldots, 1/\rho_m)$, depending on arbitrary parameters $\rho_i > 0$, $\mu_i \in \mathbb{R}$, $i = 1, \ldots, m$. Denoting the "differentiation" operators by $\mathfrak{D} = D_{(\rho_i),(\mu_i)}$, and by $\mathfrak{L} = L_{(\rho_i),(\mu_i)}$ the corresponding "integrations" (operators right inverse to $\mathfrak{D}$), we first observe that $\mathfrak{D}$ and $\mathfrak{L}$ can be considered as operators of the generalized fractional calculus, respectively—as generalized fractional "derivatives" and "integrals." A solution of the homogeneous ODE of this kind,

$$\mathfrak{D}y(z) = \lambda y(z), \quad \lambda \neq 0, \quad 0 < |z| < \infty,$$

is the recently introduced "multi-index Mittag-Leffler function" $E_{(1/\rho_i),(\mu_i)}(\lambda z)$. We find a Poisson-type integral transformation $\mathcal{P}$ (generalizing the classical Poisson integral formula) that maps the $\cos_m$-function into the multi-index Mittag-Leffler function, and also transforms the simpler differentiation and integration operators of integer order $m > 1$: $D^m = (d/dz)^m$ and l^m (the m-fold integration) into the operators $\mathfrak{D}$ and $\mathfrak{L}$. Thus, from the known solution of the Volterra-type integral equation with the m-fold integration l^m, via $\mathcal{P}$ as a transformation (transmutation) operator, we find the corresponding solution of

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the integral equation $y(z) - \lambda\mathfrak{L}(z) = f(z)$. Then, a solution of the fractional multi-order differential equation $\mathfrak{D}y(z) - \lambda y(z) = f(z)$ comes out, in an explicit form, as a series of integrals involving Fox's H-functions. For each particularly chosen R.H.S. function $f(z)$, such a solution can be evaluated as an H-function. Special cases of the equations considered here, lead to solutions in terms of the Mittag-Leffler, Bessel, Struve, Lommel and hyper-Bessel functions, and some other known generalized hypergeometric functions.

Keywords: Fractional order differential and integral equations; Operators of generalized fractional calculus; Mittag-Leffler function; Fox's H-function; Method of transmutations

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1. Introduction

The equations

$$y(z) - \lambda \int_a^z K(z,t) y(t)\, dt = f(z), \tag{1.1}$$

where $f(z)$, $K(z,t)$ are given functions, λ is a parameter and $y(z)$ is the sought solution, are called *Volterra integral equations of second kind*.

Singular integral equations of such kind arise very often in solving various problems of mathematical physics, especially that describing physical processes with after-effects (see, e.g., Gorenflo and Vessella [1]).

Fractional integral and differintegral equations involving Riemann–Liouville (R–L) integrals and/or the R–L derivatives of arbitrary (fractional) order $\delta > 0$ have been solved explicitly by various authors, such as Gorenflo and Vessella [1], Hille and Tamarkin [2], Ross and Sachdeva [3], Samko et al. [4], Srivastava and Bushman [5], etc. The solutions of the first kind Volterra integral equations $l^\delta y(z) = f(z)$ are well known. Abel (1823) was the first to solve effectively such an equation with $\delta = 1/2$ (called now *Abel integral equation*) by means of a fractional calculus operator (see [4, Ch. 1, §2.1]), thus giving good motivation for further development of this topic.

The R–L fractional integral equation of the second kind,

$$\tilde{y}(z) - \lambda l^\delta \tilde{y}(z) = \tilde{f}(z), \tag{1.2}$$

has been solved by many authors using different techniques (such as Hille and Tamarkin [2] and Ross and Sachdeva [3]), but all of them leading to the solution involving the Mittag-Leffler (M-L) function:

$$\tilde{y}(z) = \tilde{f}(z) + \lambda \int_0^z (z-t)^{\delta-1} E_{\delta,\delta}\bigl[\lambda(z-t)^\delta\bigr] \tilde{f}(t)\, dt. \tag{1.3}$$

The solution of the Cauchy problem for the R–L fractional differential equation

$$\begin{cases} D^{\delta}\tilde{y}(z) - \lambda\tilde{y}(z) = \tilde{f}(z), \\ D^{\delta-j}\tilde{y}(z)|_{z=0} = b_j, \quad j = 1, 2, \ldots, n; \quad n-1 < \delta \leqslant n, \end{cases} \tag{1.4}$$

is also expressed in terms of the M-L function [4, Examples 42.1, 42.2]:

$$\begin{aligned} \tilde{y}(z) = {} & \sum_{j=1}^{n} b_j z^{\delta-j} E_{\delta,1+\delta-j}\big(\lambda z^{\delta}\big) \\ & + \int_0^z (z-t)^{\delta-1} E_{\delta,\delta}\big[\lambda(z-t)^{\delta}\big]\tilde{f}(t)\,dt. \end{aligned} \tag{1.5}$$

Recently, new various types of differential and integral equations of fractional order have been solved, inspired by problems appearing in practice; see some details in Samko et al. [4], Podlubny [6], etc. Examples of such kinds of equations, solved in terms of the M-L functions, are

$$\begin{aligned} & D^{\delta} y(z) - \lambda l^{\nu} y(z) = f(z), \qquad z^{-\beta\delta} D_{\beta}^{\alpha,\delta} - \lambda z^{\beta\nu} I_{\beta}^{\mu,\nu} y(z) = f(z), \\ & \quad \delta > 0, \quad \nu > 0, \end{aligned}$$

which involve both fractional integrals and derivatives (in Riemann–Liouville or Erdélyi–Kober sense; see Vu and Al-Saqabi [7] and Kiryakova and Al-Saqabi [8]).

Ordinary differential equations and Volterra second kind integral equations involving some operators of the generalized fractional calculus, or other classes of generalized integrations and differentiations, have been also investigated.

Typical examples of generalized differentiation and integration operators of arbitrary order $m > 1$, although yet integer, are the hyper-Bessel differential operators B and hyper-Bessel integral operators L, appearing in various problems of applied analysis and mathematical physics (Dimovski [9,10], Kiryakova [11, Ch. 3]). The equations of the form

$$\begin{aligned} & Ly(z) = f(z), \qquad By(z) = f(z), \\ & y(z) - \lambda Ly(z) = f(z), \qquad By(z) - \lambda y(z) = f(z), \end{aligned} \tag{1.6}$$

have been explicitly solved in terms of the hyper-Bessel functions and Meijer's G-functions, or integral operators of them, either by using the theory of special functions or by using the transmutation method (see, e.g., Dimovski and Kiryakova [12], Kiryakova [11, Ch. 3], Kiryakova and McBride [13]).

We shall observe that equations like (1.2), (1.4), (1.6) and their solutions can be obtained as special cases of the problems we solve explicitly in this paper.

Namely, we consider fractional multi-order integral equations $y(z) - \lambda\mathfrak{L}y(z) = f(z)$, $\mathfrak{L}$ being a generalized fractional integration operator involving an H-function, and initial value problems for the corresponding fractional multi-order differential equations of the form

$$
\begin{aligned}
&\mathfrak{D}y(z) - \lambda y(z) \\
&\quad = z^{-1} \prod_{i=1}^{m} \bigl(z^{1+(1-\mu_i)\rho_i} D_{z^{\rho_i}}^{1/\rho_i} z^{(\mu_i-1)\rho_i}\bigr) y(z) - \lambda y(z) = f(z).
\end{aligned} \tag{1.7}
$$

For the aims of our paper, we use the so-called *transmutation method, or method of similarity, transformations method*. The essence of this method lies in solving some new and complicated problems by their reduction to well-known or simpler ones, by means of suitable "translators" (transformations, transmutations, similarity operators). In a narrow sense it originates from the works of Delsarte and Lions (see [14]) and has been widely used in mathematical analysis, and mainly for solving differential equations, by many authors (see, for example, Hearsh [15]). However, some authors, like Dimovski [10], have used "similarity" operators in the following wider sense, and also for the purposes of operational and convolutional calculi, theory of special functions, etc.

An isomorphism $T : \widetilde{X} \mapsto X$ of a linear space $\widetilde{X}$ into another linear space X is said to be a *similarity (operator)* from a linear operator $\widetilde{L} : \widetilde{X} \mapsto \widetilde{X}$ to a linear operator $L : X \mapsto X$, if $T\widetilde{L} = LT$ holds in X. We say then that *the operator $\widetilde{L}$ is similar to L under the similarity T*, or that *T transmutes $\widetilde{L}$ into L*.

2. Preliminaries

First, we recall briefly the notations and definitions for *some classes of special functions* that are basically used throughout this paper.

By *Fox's H-function* we mean a generalized hypergeometric function, defined by means of the Mellin–Barnes type contour integral

$$
\begin{aligned}
&H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] \\
&\quad = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\prod_{k=1}^{m} \Gamma(b_k - sB_k) \prod_{j=1}^{n} \Gamma(1 - a_j + sA_j)}{\prod_{k=m+1}^{q} \Gamma(1 - b_k + sB_k) \prod_{j=n+1}^{p} \Gamma(a_j - sA_j)} z^s \, ds,
\end{aligned} \tag{2.1}
$$

$z \neq 0$, where $\mathcal{C}'$ is a suitable contour in $\mathbb{C}$, the orders (m, n, p, q) are integers $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$ and the parameters $a_j \in \mathbb{R}$, $A_j > 0$, $j = 1, \ldots, p$, $b_k \in \mathbb{R}$, $B_k > 0$, $k = 1, \ldots, q$, are such that $A_j(b_k + l) \neq B_k(a_j - l' - 1)$, $l, l' = 0, 1, 2, \ldots$. For various type of contours and conditions for existence and *analyticity of the function* (2.1) *in disks* $\Delta_R = \{|z| < R\} \subset \mathbb{C}$, $R = \prod_{j=1}^{p} A_j^{-A_j} \times \prod_{k=1}^{q} B_k^{B_k} > 0$, asymptotic expansions as $|z| \to 0$ and $|z| \to \infty$, account of the basic properties and examples of (2.1), the reader is referred to [11, App.], [16–18], etc.

When $A_1 = \cdots = A_p = 1$, $B_1 = \cdots = B_q = 1$, (2.1) turns into the more popular *Meijer's G-function* (see [11,17], [19, Vol. 1, Ch. 5]):

$$G^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_j)_1^p\\(b_k)_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int\limits_{C'} \frac{\prod_{k=1}^m \Gamma(b_k-s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{k=m+1}^q \Gamma(1-b_k+s)\prod_{j=n+1}^p \Gamma(a_j-s)} z^s\,ds. \tag{2.2}$$

The G- and H-functions encompass almost all the elementary and special functions and this makes the knowledge of them very useful. For example, the ${}_pF_q$*-generalized hypergeometric functions* [19, Vol. 1] are Meijer's G-functions:

$$\begin{aligned}&{}_pF_q(a_1,a_2,\dots,a_p;b_1,b_2,\dots,b_q;z)\\&\quad=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{z^k}{k!} &(2.3)\\&\quad=\left[\frac{\prod_{i=1}^q\Gamma(b_i)}{\prod_{j=1}^p\Gamma(a_j)}\right]G^{1,p}_{p,q+1}\left[-z\,\middle|\,\begin{matrix}1-a_1,\dots,1-a_p\\0,1-b_1,\dots,1-b_q\end{matrix}\right], &(2.3')\end{aligned}$$

and so are *all their special cases*, such as the Bessel functions, the Gauss ${}_2F_1$- and Tricomi ${}_1F_1$- hypergeometric functions, the classical orthogonal polynomials, many elementary functions, etc. Let us note that the series (2.3) converges (and the corresponding G-function is analytical) for all $|z|<\infty$, if $p\leqslant q$ and for $|z|<1$, if $p=q+1$.

For the aims of this paper, we need to emphasize two special cases of (2.3). The *hyper-Bessel function* (introduced by Delerue [20]) is a multi-index analogue of the Bessel function (that follows for $m=1$),

$$J^{(m)}_{\nu_1,\nu_2,\dots,\nu_m}(z)=\frac{(z/m+1)^{\nu_1+\dots+\nu_m}}{\Gamma(\nu_1+1)\dots\Gamma(\nu_m+1)}j^{(m)}_{\nu_1,\dots,\nu_m}(z), \tag{2.4}$$

$$\text{where}\quad j^{(m)}_{\nu_1,\dots,\nu_m}(z)={}_0F_m\big((\nu_k+1)_1^m;-(z/m+1)^{m+1}\big), \tag{2.4'}$$

is called a "normalized" hyper-Bessel function, and its series is convergent for all $|z|<\infty$ (i.e., an entire function).

Like the cosine and the other trigonometric functions that follow from the Bessel function, similarly from (2.4) one obtains the *generalized cosine function of order* m:

$$\begin{aligned}\tilde{y}(z)=\cos_m(z)&=\sum_{k=0}^{\infty}\frac{(-1)^k z^{mk}}{(mk)!}\\&={}_0F_{m-1}\big((k/m)_1^{m-1};-(z/m)^m\big),\end{aligned} \tag{2.5}$$

satisfying the simplest m-order ODE, $(d/dz)^m\tilde{y}(z)=-\tilde{y}(z)$, and initial conditions $\tilde{y}(0)=1,\ \tilde{y}^{(j)}(0)=0,\ j=1,\dots,m-1$ (see [19, Vol. 3]).

The *Wright* (${}_p\Psi_q$-) *generalized hypergeometric functions*, from which the ${}_pF_q$-functions follow only when $A_1=\dots=A_p=B_1=\dots=B_q=1$:

$$
\begin{aligned}
&{}_p\Psi_q\left[\begin{matrix}(a_1,1),\dots,(a_p,1)\\(b_1,1),\dots,(b_q,1)\end{matrix}\middle| z\right]\\
&\quad=\left[\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{j=1}^{p}\Gamma(a_j)}\right]^{-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z),
\end{aligned}
$$

give examples of H-functions that, in general, are *not reducible* to G-functions:

$$
\begin{aligned}
&{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}\middle| z\right]\\
&\quad=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\dots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\dots\Gamma(b_q+kB_q)}\frac{\sigma^k}{k!}\\
&\quad=H_{p,q+1}^{1,p}\left[-z\middle|\begin{matrix}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)\end{matrix}\right].
\end{aligned}
\tag{2.6}
$$

A typical special case of (2.6) is the *Mittag-Leffler function* (see [19, Vol. 3], [21,22]), $\rho>0,\ \mu>0$:

$$
E_{1/\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k/\rho)}
\tag{2.7}
$$

$$
={}_1\Psi_1\left[\begin{matrix}(1,1)\\(\mu,1/\rho)\end{matrix}\middle| z\right]=H_{1,2}^{1,1}\left[-z\middle|\begin{matrix}(0,1)\\(0,1),(1-\mu,1/\rho)\end{matrix}\right],
\tag{2.7$'$}
$$

an entire function of order ρ. Observe that for irrational index $\rho>0$, it satisfies a differential equation of fractional order $1/\rho$.

Next, we need to recall *some notions of the fractional calculus*, the theory of the operators for integration and differentiation of arbitrary (fractional) order and their recent generalizations.

The *Riemann–Liouville* (R–L) *operator of integration* of order $\delta>0$ is defined as

$$
\begin{aligned}
l^\delta f(z)&=\frac{1}{\Gamma(\delta)}\int_0^z(z-\zeta)^{\delta-1}f(\zeta)\,d\zeta\\
&=z^\delta\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}f(z\sigma)\,d\sigma,
\end{aligned}
\tag{2.8}
$$

and, for a negative order, it is known as the *R–L fractional derivative* $l^{-\delta}:=D^\delta,\ \delta>0$,

$$
D^\delta f(z)=\begin{cases}(d/dz)^n l^{n-\delta}f(z), & n:=[\delta]+1,\text{ if }\delta\text{ noninteger},\\ f^{(n)}(z), & \text{if }\delta=n,\text{ integer}.\end{cases}
\tag{2.9}
$$

We shall use also the denotations

$$
\begin{aligned}
&I^{\delta}_{z^{\beta}} f(z) := \left[I^{\delta}_{w} f\left(w^{1/\beta}\right)\right]_{w=z^{\beta}},\\
&D^{\delta}_{z^{\beta}} f(z) := \left[D^{\delta}_{w} f\left(w^{1/\beta}\right)\right]_{w=z^{\beta}}, \quad \beta > 0.
\end{aligned}
\tag{2.10}
$$

The frequent use of operators like (2.10), combined with power weights associated to the integrand, explains the recent popularity of the following modification of the R–L fractional integrals and derivatives. The *Erdélyi–Kober* (*E–K*) *operator of fractional integration* is defined for order $\delta > 0$, weight $\gamma \in \mathbb{R}$ and parameter $\beta > 0$ as

$$
\begin{aligned}
I^{\gamma,\delta}_{\beta} y(z) &= \left[\left(z^{-(\gamma+\delta)} I^{\delta} z^{\gamma}\right) y\left(z^{1/\beta}\right)\right]_{z \to z^{\beta}}\\
&= \frac{z^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^z \left(z^{\beta} - t^{\beta}\right)^{\delta-1} t^{\beta\gamma} y(t)\, d\left(t^{\beta}\right)\\
&= \frac{1}{\Gamma(\delta)} \int_0^1 (1-\sigma)^{\delta-1} \sigma^{\gamma} y\left(z\sigma^{1/\beta}\right) d\sigma,
\end{aligned}
\tag{2.11}
$$

and the *E–K fractional derivative* is denoted by (see [11, Ch. 2])

$$
D^{\gamma,\delta}_{\beta} y(z) = \left[\left(z^{-\gamma} D^{\delta} z^{\gamma+\delta}\right) y\left(z^{1/\beta}\right)\right]_{z \to z^{\beta}}.
\tag{2.12}
$$

For the whole details of the theory of the classical fractional calculus and its applications, we refer to the encyclopaedic book of Samko et al. [4].

In [23,24], Kalla introduced the notion of the *generalized operators of fractional integration*

$$
\mathcal{R} f(z) = z^{-(\gamma+1)} \int_0^z \Phi(\zeta/z) \zeta^{\gamma} f(\zeta)\, d\zeta = \int_0^1 \Phi(\sigma) \sigma^{\gamma} f(z\sigma)\, d\sigma,
\tag{2.13}
$$

where the kernel-function Φ can be arbitrary so the integral makes sense in suitable functional space for $f(z)$, and studied some of their basic properties when Φ is the Gauss, Meijer's or Fox's hypergeometric function.

The *generalized fractional calculus*, in the book of Kiryakova [11], is based on commutative *compositions of E–K operators* (2.11), (2.12),

$$
\begin{aligned}
I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) &= \left[\prod_{k=1}^{m} I^{\gamma_k,\delta_k}_{\beta_k}\right] f(z)\\
&= \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-\sigma_k)^{\delta_k-1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)}\right]\\
&\quad \times f\left(z\sigma_1^{1/\beta_1} \ldots \sigma_m^{1/\beta_m}\right) d\sigma_1 \ldots d\sigma_m,
\end{aligned}
\tag{2.14$'$}
$$

but represented in the single integral form of (2.13), by means of suitable H- or G-functions as kernels. Namely, let $m \geqslant 1$ be an integer, $\delta_k \geqslant 0$, $\gamma_k \in \mathbb{R}$, $\beta_k > 0$, $i = 1, \dots, m$, and consider $\delta = (\delta_1, \dots, \delta_m)$ as a *multi-order of fractional integration*. The integral operators defined as

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = \begin{cases} \int_0^1 H_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k + 1 - 1/\beta_k, 1/\beta_k)_1^m \\ (\gamma_k + 1 - 1/\beta_k, 1/\beta_k)_1^m \end{matrix}\right] f(z\sigma)\, d\sigma, \\ \quad \text{if } \sum_{k=1}^m \delta_k > 0, \\ f(z), \quad \text{if } \delta_1 = \delta_2 = \cdots = \delta_m = 0, \end{cases} \tag{2.14}$$

are said to be *multiple* (m*-tuple*) *Erdélyi–Kober fractional integration operators*, and more generally, all the operators of the form

$$\mathcal{I} f(z) = z^{\delta_0} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) \quad \text{with } \delta_0 \geqslant 0, \tag{2.15}$$

are called briefly *generalized* (m*-tuple*) *fractional integrals*.

By a *generalized fractional derivative*, we mean an integro-differential (or differential) operator of the form

$$\mathcal{D} f(z) = z^{-\delta_0} D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}, \quad \delta_0 \geqslant 0, \tag{2.16}$$

where the operators corresponding to (2.14), namely $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = \prod_{k=1}^m D_{\beta_k}^{\gamma_k,\delta_k}$, are defined by means of explicit differintegral expressions. To this end we denote

$$D_\eta = \left[\prod_{k=1}^m \prod_{j=1}^{\eta_k} \left(\frac{1}{\beta_j} x \frac{d}{dx} + \gamma_k + j\right)\right],$$

$$\eta_k = \begin{cases} [\delta_k] + 1, & \text{if } \delta_k \text{ noninteger}, \\ \delta_k, & \text{if } \delta_k \text{ integer}, \end{cases} \qquad k = 1, \dots, m,$$

and define

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) := D_\eta I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(z). \tag{2.17}$$

The theory of the generalized fractional integrals and derivatives (2.15), (2.16) has been developed in Kiryakova [11], together with various applications. As simpler cases, when $\beta_k = \beta > 0$, $k = 1, \dots, m$, are equal, generalized fractional integrals and derivatives involving the kernel $G_{m,m}^{m,0}$-function are studied and applied in Chapters 1, 3, 4, instead of the H-function case from [11, Ch. 5], which was first introduced in Kalla and Kiryakova [25].

Further, Srivastava et al. [26] introduced *modifications, more general than operators* (2.14), (2.17) when the kernel H-function in (2.14) can contain different second parameters β_k, $\lambda_k > 0$, $k = 1, \dots, m$, in the upper and lower rows:

$$\mathcal{I} = I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)} f(z)$$
$$= \begin{cases} \int_0^1 H_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+\delta_k+1-1/\beta_k, 1/\beta_k)_1^m \\ (\gamma_k+1-1/\lambda_k, 1/\lambda_k)_1^m \end{matrix}\right] f(z\sigma)\,d\sigma, \\ \quad \text{if } \sum_{k=1}^m \delta_k > 0, \\ f(z), \quad \text{if } \delta_k = 0,\ \lambda_k = \beta_k,\ k = 1,\dots,m, \end{cases} \tag{2.18}$$

and $\sum_1^m (1/\lambda_k) \geqslant \sum_1^m (1/\beta_k)$ is supposed. We call (2.18) by the same name, *generalized fractional integrals*.

Let us note that if $\lambda_k = \beta_k,\ k = 1,\dots,m$, operators (2.18) and (2.14) coincide:

$$I_{(\beta_k),(\beta_k),m}^{(\gamma_k),(\delta_k)} = I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}. \tag{2.19}$$

A chain of operational rules, analogues of the rules of the classical fractional calculus, have been developed for operators (2.14), (2.17) in Kiryakova [11, Ch. 5], and for (2.18) and their corresponding fractional derivatives in Srivastava et al. [26]. The above-mentioned operators have been studied in various functional spaces, as weighted spaces of Lebesgue integrable, continuous or analytic functions.

For the purposes of this paper, it is most convenient to consider *analytic functions $f(z)$ in simply connected regions of the complex z-plane containing the origin*. In this case, multiplicities like those of $(z-\zeta)^{\delta-1}$ in the R–L integral (2.8) are removed by requiring $\log(z-\zeta)$ to be real for $(z-\zeta) > 0$. In the same style, *unique branches* of z^δ, z^μ, etc., are determined in all the cases when multiplicities appear in the definitions of the generalized fractional integration or differentiation operators on classes of analytic functions.

Denote by $\mathcal{H}(\Omega)$ *the space of analytic* (*and single valued*) *functions* in a complex domain $\Omega \subset \mathbb{C}$, starlike with respect to the origin $z = 0$ (therefore, a simply connected one). For $\alpha \in \mathbb{R}$, we consider the classes

$$\mathcal{H}_\alpha(\Omega) = \left\{ f(z) = z^p \tilde{f}(z);\ p \geqslant \alpha,\ \tilde{f}(z) \in \mathcal{H}(\Omega) \right\},$$
$$\mathcal{H}_0(\Omega) := \mathcal{H}(\Omega). \tag{2.20}$$

In particular, Ω can be a disk $\Delta_R = \{|z| < R\}$, $R > 0$, and then the functions $f(z)$ are given by their power series

$$f(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}_\alpha(\Delta_R), \tag{2.21}$$

absolutely convergent in Δ_R with $R = \{\limsup_{n\to\infty} \sqrt{n}|a_n|\}^{-1}$.

It is easy to see that the operators of the fractional calculus (classical and generalized) are well defined on any $\mathcal{H}_\alpha(\Omega)$ under suitable conditions on their parameters. For example, the *E–K operators and their generalizations* $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$, $I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}$ *map $\mathcal{H}_\alpha(\Omega)$ into itself*, if $\alpha \geqslant \max_{1\leqslant k\leqslant m}\{-\beta_k(\gamma_k+1)\}$ for (2.14), and if $\alpha \geqslant \max_{1\leqslant k\leqslant m}\{-\lambda_k(\gamma_k+1)\}$ for (2.18).

Lemma 1. *Let the conditions $\gamma_k > -1-\alpha/\beta_k$, $k=1,\dots,m$, be satisfied. Then* (2.14), $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ *maps $\mathcal{H}_\alpha(\Omega)$ into itself, preserving the power functions up to a constant multiplier*:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\{z^p\} = c_p z^p, \quad p \geqslant \alpha,$$

$$\text{with} \quad c_p = \prod_{k=1}^{m} \frac{\Gamma(\gamma_k+1+p/\beta_k)}{\Gamma(\gamma_k+\delta_k+1+p/\beta_k)} > 0. \tag{2.22}$$

Hence, the image of a power series (2.21) *from $\mathcal{H}_\alpha(\Delta_R)$ is given by*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = z^\alpha \sum_{n=0}^{\infty} \left\{ a_n \prod_{k=1}^{m} \frac{\Gamma(\gamma_k+1+(n+\alpha)/\beta_k)}{\Gamma(\gamma_k+\delta_k+1+(n+\alpha)/\beta_k)} \right\} z^n, \tag{2.22$'$}$$

having the same radius of convergence $R>0$ as (2.21) *and the same signs of the coefficients. More generally, each generalized fractional integral $\mathcal{I}f(z) = z^{\delta_0} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z)$, $\delta \geqslant 0$, maps $\mathcal{H}_\alpha(\Omega)$ into $\mathcal{H}_{\alpha+\delta_0}(\Omega) \subseteq \mathcal{H}_\alpha(\Omega)$.*

For the proof see Kiryakova [11, Theorem 5.5.2] and Kiryakova and Srivastava [27, Theorem 1]. A similar proposition holds for the modifications $I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}$ in (2.18), provided $\gamma_k > -1-\alpha/\lambda_k$, $k=1,\dots,m$.

3. Multi-index M-L functions and fractional multi-order operators

Recently, the interest in the Mittag-Leffler functions and their popularity have increased considerably in view of their important role and applications in fractional calculus and related integral and differential equations of fractional order, solutions of problems in control theory (fractional order control systems, fractional order controllers), continuum mechanics, fractional viscoelastic models, diffusion theory, fractals, etc. (see, for example, Podlubny [6] and many other recent books, surveys and articles).

Given this background, and in order to extend our studies on the hyper-Bessel functions (2.4) (multi-index analogues of the Bessel function, related to Bessel-type differential equations of arbitrary order $m>1$), we have recently introduced and studied the following *multi-index generalizations of the Mittag-Leffler function*.

Definition 1. Let $m>1$ be an integer, $\rho_1>0,\dots,\rho_m>0$ and $\mu_1,\dots,\mu_m$ be arbitrary real numbers. By means of the “multi-indices” (ρ_i), (μ_i), we define the *multi-index Mittag-Leffler functions* (*m-M-L functions*)

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty} \varphi_k z^k = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1) \ldots \Gamma(\mu_m + k/\rho_m)}. \tag{3.1}$$

It has been proved [28] that (3.1) are entire functions of order $\rho := (1/\rho_1 + \cdots + 1/\rho_m)^{-1}$ and type $\sigma > 1$ (in the classical case $m = 1$: $\sigma = 1$), and are representable as Wright's generalized hypergeometric functions and H-functions, namely

$$E_{(1/\rho_i),(\mu_i)}(z) = {}_1\Psi_m\left[\begin{matrix}(1,1)\\ (\mu_i, 1/\rho_i)_1^m\end{matrix}\,\middle|\, z\right] = H^{1,1}_{1,m+1}\left[-z\,\middle|\,\begin{matrix}(0,1)\\ (0,1), (1-\mu_i, 1/\rho_i)_1^m\end{matrix}\right]. \tag{3.2}$$

From the above, the corresponding Mellin–Barnes type contour integral representation, asymptotic behaviour as $|z| \to 0, \infty$, fractional integral and differential relations involving the E–K operators (2.11), and other properties are found.

For the first time, generalizations of the M-L functions, of the kind of (3.1), were studied by Dzrbashjan [29], for $m = 2$. As special cases of $E_{(1/\rho_i),(\mu_i)}(z)$, we can derive the classical M-L functions, the Bessel and hyper-Bessel functions, the Bessel–Maitland, Struve and Lommel functions, the ${}_1F_m$-functions, etc. The classical M-L functions (2.7), on their side, are natural generalizations of the exponential and trigonometrical functions (as solutions of ODE of integer orders), and incorporate also the error functions, the incomplete gamma functions etc.

Therefore, the role of the multi-index M-L functions, as solutions of rather general classes of differential and integral equations of fractional multi-order $(1/\rho_1, 1/\rho_2, \ldots, 1/\rho_m)$, is important. Here we describe the corresponding generalized differential and integral operators $\mathfrak{D}$, $\mathfrak{L}$ related to them, and in the next sections we deal with the task of finding solutions of classes of equations involving these operators.

Definition 2. Let $f(z)$ be an analytic function in a disk $\Delta_R = \{|z| < R\}$ and $\rho_i > 0,\ \mu_i \in \mathbb{R},\ i = 1, \ldots, m$, be arbitrary parameters. The operators in $\mathcal{H}(\Delta_R)$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

$$\mapsto \begin{cases} \mathfrak{D} f(z) = D_{(\rho_i),(\mu_i)} f(z) \\ \qquad = \sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\ldots\Gamma(\mu_m+\frac{k}{\rho_m})}{\Gamma(\mu_1+\frac{k-1}{\rho_1})\ldots\Gamma(\mu_m+\frac{k-1}{\rho_m})} z^{k-1}, \\ \mathfrak{L} f(z) = L_{(\rho_i),(\mu_i)} f(z) \\ \qquad = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\ldots\Gamma(\mu_m+\frac{k}{\rho_m})}{\Gamma(\mu_1+\frac{k+1}{\rho_1})\ldots\Gamma(\mu_m+\frac{k+1}{\rho_m})} z^{k+1} \end{cases} \tag{3.3}$$

are called *fractional multi-order differentiation and integration operators* (with respect to multi-index M-L function (3.1)).

In a more precise terminology, these are *Gelfond–Leontiev* (*G–L*) *operators of generalized differentiation and integration* with respect to the entire function $\varphi(z) = E_{(1/\rho_i),(\mu_i)}(z)$, and in our previous works they were named "multiple Dzrbashjan–Gelfond–Leontiev (D–G–L) operators" (cf. [28, (5), (6), (35)]). The simplest case, of G–L operators with respect to $\varphi(z) = \exp(z)$, evidently gives the classical differentiation and integration. But for the purposes of this paper, the shorter notion used in Definition 2 seems more suitable.

Let us note that $\mathfrak{D}\mathfrak{L} f(z) = D_{(\rho_i),(\mu_i)} L_{(\rho_i),(\mu_i)} f(z) = f(z)$ in $\mathcal{H}(\Delta_R)$ and the coincidence of the radii of convergence of $f(z)$ and series (3.3) follows easily by the Cauchy–Hadamard formula and the asymptotic estimation of the Γ-function multipliers, in a way similar to [11, Theorem 5.5.2; 27].

The series representations in (3.3) can be analytically continued for analytic functions in starlike domains $\Omega \supset \Delta_R$ by means of single integral or differintegral expressions, as special cases of the operators (2.17), (2.14) of the generalized fractional calculus. *This also justifies the names given in Definition 2*. Namely, let $\Delta_R \subset \Omega,\ \mu_i \geqslant 0,\ i = 1, \ldots, m$; i.e., $\alpha = \max_{1\leqslant k\leqslant m}\{-\mu_k\rho_k\} \leqslant 0$. Then (see [28, Theorem 4.2]), the multi-order fractional integration operator (3.3) can be analytically continued from $\mathcal{H}(\Delta_R)$ into $\mathcal{H}_\alpha(\Omega)$, and therefore also in $\mathcal{H}(\Omega)$, as a generalized fractional integral of form (2.14) (see Lemma 1):

$$\begin{aligned} \mathfrak{L} f(z) &= L_{(\rho_i),(\mu_i)} f(z) \\ &= z \int_0^1 H_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\mu_i, 1/\rho_i)_1^m \\ (\mu_i - 1/\rho_i, 1/\rho_i)_1^m \end{matrix}\right] f(z\sigma)\, d\sigma \\ &= z I_{(\rho_i),m}^{(\mu_i-1),(1/\rho_i)} f(z). \end{aligned} \tag{3.4}$$

The fractional multi-order differentiation $D_{(\rho_i),(\mu_i)}$ in (3.3) also has a differintegral representation, in terms of (2.17): for analytic functions in $\mathcal{H}(\Omega) \supset \mathcal{H}(\Delta_R)$,

$$\begin{aligned} \mathfrak{D} f(z) &= D_{(\rho_i),(\mu_i)} f(z) \\ &= z^{-1} D_{(\rho_i),m}^{(\mu_i-1-1/\rho_i),(1/\rho_i)} f(z) - \left[\prod_{i=1}^m \frac{\Gamma(\mu_i)}{\Gamma(\mu_i - 1/\rho_i)}\right] \frac{f(0)}{z}. \end{aligned} \tag{3.5}$$

One can give another, more explicit form to the fractional multi-order differentiation operator (3.3)–(3.5), using denotations in terms of the R–L fractional derivatives (2.10) (i.e., the decomposition property of the generalized fractional derivatives (2.17) into E–K fractional derivatives (2.12); see [11]):

$$\begin{aligned}\mathfrak{D} f(z) &= D_{(\rho_i),(\mu_i)} f(z) \\ &= z^{-1} \prod_{i=1}^{m} \left(z^{1+(1-\mu_i)\rho_i} D_{z^{\rho_i}}^{1/\rho_i} z^{(\mu_i-1)\rho_i} \right) f(z). \end{aligned} \tag{3.6}$$

Thus, the operators $\mathfrak{D} = D_{(\rho_i),(\mu_i)}$ can be seen also as *fractional order analogues* of the hyper-Bessel differential operators (of *integer* multi-order $(1, 1, \dots, 1)$),

$$\begin{aligned} &B = z^{\alpha_0} \frac{d}{dz} z^{\alpha_1} \frac{d}{dz} z^{\alpha_2} \dots \frac{d}{dz} z^{\alpha_m} = z^{-\beta} \prod_{i=1}^{m} \left(z^{-\beta\nu_i+1} \frac{d}{dz} z^{\beta\nu_i} \right), \\ &m > 1, \quad \beta > 0, \end{aligned} \tag{3.7}$$

studied by Dimovski [9,10], Dimovski and Kiryakova [30], and Kiryakova [11, Ch. 3].

It can be easily seen, by the definitions of $E_{(1/\rho_i),(\mu_i)}(z)$, $L_{(\rho_i),(\mu_i)}$ and $D_{(\rho_i),(\mu_i)}$, that

$$\begin{aligned}\mathfrak{L} E_{(1/\rho_i),(\mu_i)}(\lambda z) &= L_{(\rho_i),(\mu_i)} E_{(1/\rho_i),(\mu_i)}(\lambda z) \\ &= \frac{1}{\lambda} E_{(1/\rho_i),(\mu_i)}(\lambda z) - \frac{1}{\lambda \prod_i \Gamma(\mu_i)}, \end{aligned} \tag{3.8}$$

and, consequently, *the multi-index Mittag-Leffler function* (3.1), (3.2) *satisfies the fractional multi-order differential equation* (of order $1/\rho := 1/\rho_1 + \cdots + 1/\rho_m > 0$):

$$\begin{aligned} &\mathfrak{D} E_{(1/\rho_i),(\mu_i)}(\lambda z) = D_{(\rho_i),(\mu_i)} E_{(1/\rho_i),(\mu_i)}(\lambda z) = \lambda E_{(1/\rho_i),(\mu_i)}(\lambda z), \\ &\lambda \neq 0. \end{aligned} \tag{3.9}$$

The latter result is a useful suggestion for our considerations in the next sections.

4. Poisson-type transformation

The term *Poisson-type transformation* comes from the well known *Poisson integral formula* [19, Vol. 2]

$$J_\nu(z) = \frac{2}{\sqrt{\pi}} \frac{(z/2)^\nu}{\Gamma(\nu + 1/2)} \int_0^{\pi/2} \cos(z \sin\varphi)(\cos\varphi)^{2\nu} \, d\varphi$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu}\int_0^1 \frac{(1-t^2)^{\nu-1/2}}{\Gamma(\nu+1/2)}\cos(zt)\,dt. \tag{4.1}$$

It represents the Bessel function as an integral transformation P_ν of the (simpler) cosine function, P_ν being an *Erdélyi–Kober operator of fractional integration* (2.11) of order $\alpha = \nu + 1/2 > 0$:

$$\begin{aligned} P_\nu f(z) &= \frac{2}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu}\int_0^1 \frac{(1-t^2)^{\nu-1/2}}{\Gamma(\nu+1/2)} f(zt)\,dt \\ &= \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu}\int_0^1 \frac{(1-\sigma)^{\nu-1/2}}{\Gamma(\nu+1/2)}\sigma^{-1/2} f\left(z\sqrt{\sigma}\right)d\sigma, \\ \nu &> -1/2. \end{aligned} \tag{4.2}$$

Developing a theory of the Bessel-type differential operators (3.7) of arbitrary order $m \geqslant 2$ and the respective integral operators L, called in common *hyper-Bessel operators* (see [11, Ch. 3]), Dimovski [9,10] introduced a far-reaching generalization of the transformation P_ν,

$$\begin{aligned} Tf(z) &= c\left(\frac{z^\beta}{\beta^m}\right)^{\lambda}\int_0^1 \cdots \int_0^1 \prod_{k=1}^m\left[\frac{(1-t_k)^{\lambda+\gamma_k-k/m}}{\Gamma(\lambda+\gamma_k-k/m+1)}t_k^{k/m-1}\right] \\ &\quad\times f\left[\frac{m}{\beta}z^{\beta/m}(t_1\ldots t_m)^{1/m}\right]dt_1\ldots dt_m, \\ \lambda &\geqslant \max_k\left(\frac{k}{m}-\gamma_k\right), \quad c = \sqrt{\frac{m}{(2\pi)^{m-1}}}\prod_{k=1}^m \Gamma(\gamma_k+1). \end{aligned}$$

He proved that T is a similarity (transmutation operator) between the m-fold integration l^m and the hyper-Bessel integral operator L, viz. $Tl^m = LT$. Later on, in the works of Dimovski and Kiryakova [30] and Kiryakova [11], the operator T was represented in terms of Meijer's G-function,

$$Tf(z) = c\left(\frac{z^\beta}{\beta^m}\right)^{\lambda}\int_0^1 G^{m,0}_{m,m}\left[\sigma\,\middle|\,\begin{matrix}(\gamma_k+\lambda)\\(k/m-1)\end{matrix}\right] f\left[\frac{m}{\beta}z^{\beta/m}\sigma^{1/m}\right]d\sigma, \tag{4.3}$$

or for shortness of denotations, if we take $\lambda = -\gamma_m = 0$, $\beta = m$, this is a generalized fractional integration operator (2.14), or product of E–K operators (2.11):

$$Tf(z) = cI^{(k/m-1),(\gamma_k-k/m+1)}_{m,m-1} f(z) = c\prod_{k=1}^{m-1} I_m^{k/m-1,\gamma_k-k/m+1} f(z). \tag{4.3$'$}$$

We call (4.3), (4.3′) a *Poisson–Dimovski transformation*. It transforms the generalized (m-order) cosine function $\cos_m$ into a hyper-Bessel function (2.4); i.e., gives the *generalized Poisson integral formula* ([11, Chs. 3, 5], [12]):

$$J^{(m-1)}_{\gamma_1,\dots,\gamma_{m-1}}(z) = c\frac{(z/m)^{\sum\gamma_k}}{\prod\Gamma(\gamma_k+1)}\,{}_0F_{m-1}\big((\gamma_m+1); -(z/m)^m\big)$$
$$= \frac{(z/m)^{\sum\gamma_k}}{\prod\Gamma(\gamma_k+1)}T\big\{\cos_m(z)\big\}. \tag{4.4}$$

In this paper, we consider a further generalization of the operators P_ν and T that transforms the $\cos_m$-function into the recently introduced multi-index Mittag-Leffler function (3.1) $E_{(1/\rho_i),(\mu_i)}(z)$ and serves as a transmutation operator (similarity) from the simpler operators of m-times differentiation/integration into the generalized differential/integral operators of fractional multi-order $\mathfrak{D}, \mathfrak{L}$. Thus, by the transmutation method, we are able to find out explicit solutions to wide classes of nonhomogeneous integral and differential equations of fractional multi-order, the main object of our study.

The hint for the form of the new Poisson-type transformation comes from the ideas and results only briefly outlined in Kiryakova [31] and Srivastava et al. [26]. Namely, in [31] we have shown that the basic classes of the known special functions, viz. the generalized hypergeometric functions ${}_pF_q$ and ${}_p\Psi_q$, can be represented by suitable operators of the generalized fractional calculus of three basic simplest (elementary) functions, depending on whether $p<q$, $p=q$ or $p=q+1$. Especially in the case $p<q$, Poisson-type formulas are used and the special functions are representable as generalized fractional integrals/derivatives of the $\cos_m$-function. The case of the ${}_pF_q$-functions has been studied in detail in [11, Chs. 3, 4] as well as in Section 3 of [31]. The more complicated case of the Wright's functions ${}_p\Psi_q$ is briefly discussed only in the last section of Kiryakova [31] and it turns out that it needs the application of the generalized fractional integrals/derivatives in the wider sense of the paper by Srivastava et al. [26]. Namely, from Theorem 4.6 in [31], one easily derives as a special case the following representation, a *Poisson-type formula for the multi-index Mittag-Leffler functions*:

$$E_{(1/\rho_i),(\mu_i)}(-z) = {}_1\Psi_m\left[\begin{matrix}(1,1)\\(\mu_i,1/\rho_i)_1^m\end{matrix}\,\middle|\,-z\right]$$
$$= \sqrt{\frac{m}{(2\pi i)^{m-1}}}\,I^{(k/m-1)_1^m,(\mu_k-k/m)_1^m}_{(\rho_k),(1),m}\big\{\cos_m\big(mz^{1/m}\big)\big\}. \tag{4.5}$$

Definition 3. The operator of the generalized fractional calculus (in sense of [26])

$$\mathcal{P}f(z) = c^* I^{(k/m-1)_1^m,(\mu_k-k/m)_1^m}_{(\rho_k),(1),m} f(mz), \qquad c^*=\sqrt{m/(2\pi i)^{m-1}}, \tag{4.6}$$

of form (2.18), is called a *generalized Poisson transformation, corresponding to the multi-index M-L function* $E_{(1/\rho_i),(\mu_i)}$ *and to the operators* $\mathfrak{D}$, $\mathfrak{L}$, since

$$E_{(1/\rho_i),(\mu_i)}(-z) = \mathcal{P}\big\{\cos_m(z^{1/m})\big\}. \tag{4.5$'$}$$

For $\mu_k \geqslant k/m,\ k = 1,\dots,m$, the transformation $\mathcal{P}$ is a fractional integration operator of positive multi-order, involving the H-function,

$$\mathcal{P}f(z) = c^* \int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix}(\mu_k - 1/\rho_k, 1/\rho_k)\\ (k/m-1, 1)\end{matrix}\right] f(mz\sigma)\,d\sigma, \tag{4.6$'$}$$

and otherwise, it contains integro-differentiation (or purely differential) operators (see the definition of the generalized fractional derivatives [26]).

Next, we establish the *role of the transformation* $\mathcal{P}$ *as a transmutation operator* between some integral and differential operators and corresponding equations.

Consider the following three "differential" operators:

$$\begin{aligned}
&D^m = \left(\frac{d}{dz}\right)^m,\ D^m_{z^{1/m}} \quad \text{of integer order } m > 1,\\
&\text{and} \quad \mathfrak{D} = D_{(\rho_i),(\mu_i)} \quad \text{(cf. (3.3), (3.5), (3.6))}\\
&\qquad \text{of fractional order } \frac{1}{\rho} = \frac{1}{\rho_1} + \cdots + \frac{1}{\rho_m}.
\end{aligned} \tag{4.7}$$

Equation ($\widetilde{\mathbf{E}}$).

$$D^m\tilde{y}(z) = \left(\frac{d}{dz}\right)^m \tilde{y}(z) = -\tilde{y}(z), \tag{$\widetilde{\mathbf{E}}$}$$

with a solution $\tilde{y}(z) = \cos_m(z)$.

Equation ($\widehat{\mathbf{E}}$).

$$\begin{aligned}
&D^m_{z^{1/m}}\hat{y}(z) = -\hat{y}(z),\\
&\text{where} \quad D^m_{z^{1/m}}\hat{y}(z) = \big[D^m_w\hat{y}(w^m)\big]_{w=z^{1/m}},
\end{aligned} \tag{$\widehat{\mathbf{E}}$}$$

with a solution $\hat{y}(z) = \cos_m(z^{1/m})$.

Equation (E).

$$\mathfrak{D}y(z) = D_{(\rho_i),(\mu_i)}y(z) = -y(z), \tag{E}$$

with a solution $y(z) = E_{(1/\rho_i),(\mu_i)}(-z)$ (cf. (3.9) with $\lambda = -1$).

The generalized Poisson formula (4.5′) means that the operator $\mathcal{P}$ transforms a solution of Eq. $(\widehat{\mathbf{E}})$ into a solution of Eq. $(\mathbf{E})$. It is easily seen that $(\widetilde{\mathbf{E}})$ and $(\widehat{\mathbf{E}})$, and their solutions, are related by the simple transformations (changes of the variable)

$$\chi : \tilde{y}(z) \mapsto \hat{y}(z) := \tilde{y}(z^{1/m}), \qquad \chi^{-1} : \hat{y}(z) \mapsto \tilde{y}(z) := \hat{y}(z^m), \tag{4.8}$$

namely

$$(\widetilde{\mathbf{E}}) \xrightarrow{\chi} (\widehat{\mathbf{E}}) \xrightarrow{\mathcal{P}} (\mathbf{E}), \quad \text{i.e.,} \quad (\widetilde{\mathbf{E}}) \xrightarrow{\mathcal{P}^*=\mathcal{P}\chi} (\mathbf{E}), \tag{4.9}$$

and

$$\begin{aligned} &\tilde{y}(z) = \cos_m(z) \xrightarrow{\chi} \hat{y}(z) = \cos_m(z^{1/m}) \xrightarrow{\mathcal{P}} y(z) = E_{(1/\rho_i),(\mu_i)}(-z), \\ &\text{i.e.,} \quad \tilde{y}(z) = \cos_m(z) \xrightarrow{\mathcal{P}^*=\mathcal{P}\chi} y(z) = E_{(1/\rho_i),(\mu_i)}(-z). \end{aligned} \tag{4.9′}$$

It happens that the same operators (4.6) and (4.8) give *relationships* (*similarities*) *between the corresponding integral operators* that are linear right inverses to the differential operators in (4.7) and satisfy zero initial conditions: the operators of m-fold integration

$$\begin{aligned} &l^m f(z) = \int_0^z \int_0^{\zeta_1} \cdots \int_0^{\zeta_{m-1}} f(\zeta_m)\, d\zeta_m = z^m \int_0^1 \frac{(1-\sigma)^{m-1}}{(m-1)!} f(z\sigma)\, d\sigma, \\ &l^m_{z^{1/m}} = \chi l^m_z \chi^{-1}, \\ &\text{and} \quad \mathfrak{L} = L_{(\rho_i),(\mu_i)} = z I^{(\mu_k-1),(1/\rho_k)}_{(1/\rho_k),m} = z I^{(\mu_k-1),(1/\rho_k)}_{(1/\rho_k),(1/\rho_k),m}, \end{aligned} \tag{4.7′}$$

the generalized fractional integral (3.4).

Lemma 2. *The generalized Poisson transformation*

$$\mathcal{P}^* = \mathcal{P}\chi \quad (cf.\ (4.6), (4.8)) \tag{4.10}$$

is a transmutation operator from the m-fold integration l^m to the generalized fractional integration operator $\mathfrak{L} = L_{(\rho_i),(\mu_i)}$, namely $\mathcal{P}^* : l^m \to \mathfrak{L}$ in the space $\mathcal{H}(\Omega)$, that is, the similarity relations hold:

$$\mathcal{P}^* l^m f(z) = L_{(\rho_i),(\mu_i)} \mathcal{P}^* f(z), \quad f \in \mathcal{H}(\Omega), \tag{4.11}$$

respectively,

$$\begin{aligned} &\mathfrak{L} = L_{(\rho_i),(\mu_i)} = \mathcal{P}^* l^m (\mathcal{P}^*)^{-1} \quad \text{and} \quad l^m = (\mathcal{P}^*)^{-1} L_{(\rho_i),(\mu_i)} \mathcal{P}^*, \\ &\quad \text{in } \mathcal{H}(\Omega). \end{aligned} \tag{4.11′}$$

Proof. According to Lemma 1, the operators $\mathcal{P}$, l^m and $\mathfrak{L}$, being of the form (2.15) (or modified), preserve the space $\mathcal{H}(\Omega)$ of functions, analytic in a starlike domain Ω containing the origin $z = 0$; the same holds true also for their products.

To prove relations (4.11), (4.11′), first we need another representation of the simple m-fold integration operator l^m, although much more complicated, but in the same form as the generalized fractional integration operators $L_{(\rho_i),(\mu_i)}$ and $\mathcal{P}^*$, so we may use the operational rules of the generalized fractional calculus for the product $\mathcal{P}^* l^m$ and to compare with the other product $L_{(\rho_i),(\mu_i)}\mathcal{P}^*$.

As in [11, p. 117, (3.3.2)] we can first put l^m in the form of a generalized fractional integral with a Meijer's G-function as a kernel, using the properties of this function [19, Vol. 1]:

$$\begin{aligned} l^m f(z) &= z^m \int_0^1 \frac{(1-\sigma)^{m-1}}{\Gamma(m)} f(z\sigma)\,d\sigma \\ &= z^m \int_0^1 G_{1,1}^{1,0}\left[\sigma \,\middle|\, \begin{matrix} m \\ 0 \end{matrix}\right] f(z\sigma)\,d\sigma \\ &= \left(\frac{z}{m}\right)^m \int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} \left(\frac{k}{m}\right)_1^m \\ \left(\frac{k}{m}-1\right)_1^m \end{matrix}\right] f(z\sigma^{1/m})\,d\sigma \\ &= m\left(\frac{z}{m}\right)^m \int_0^1 G_{m,m}^{m,0}\left[\tau^m \,\middle|\, \begin{matrix} \left(\frac{k-1}{m}+1\right) \\ \left(\frac{k-1}{m}\right) \end{matrix}\right] f(z\tau)\,d\tau. \end{aligned}$$

Then, due to the relation (see [18] or, e.g., [11, p. 345, (E.6′)])

$$G_{p,q}^{m,n}\left[z^{1/c} \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right] = c H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,c), (a_2,c), \ldots, (a_p,c) \\ (b_1,c), (b_2,c), \ldots, (b_q,c) \end{matrix}\right],$$

with $c = 1/m > 0$, we obtain

$$\begin{aligned} l^m f(z) &= \left(\frac{z}{m}\right)^m \int_0^1 H_{m,m}^{m,0}\left[\tau \,\middle|\, \begin{matrix} \left(\frac{k-1}{m}+1, \frac{1}{m}\right)_1^m \\ \left(\frac{k-1}{m}, \frac{1}{m}\right)_1^m \end{matrix}\right] f(z\tau)\,d\tau \\ &= \left(\frac{z}{m}\right)^m I_{(m),(m),m}^{(k/m-1),(1)} f(z), \end{aligned} \tag{4.12}$$

as a generalized fractional integral of the form (2.18) in sense of Srivastava et al. [26].

Let us note also that a simple transformation like $\Xi : f(z) \mapsto f(z^\omega)$, $\omega > 0$, is a similarity between generalized fractional integration (differentiation) operators of form (2.18) with β_k, λ_k and $\omega\beta_k, \omega\lambda_k$, respectively, namely (see Lemma 1.3.3 in [11]):

$$\Xi\left[z^\alpha I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}\right] f(z) = \left[z^{\omega\alpha} I_{(\omega\beta_k),(\omega\lambda_k),m}^{(\gamma_k),(\delta_k)}\right] \Xi f(z). \tag{4.13}$$

Then, using (4.13) for $\Xi = \chi$ with $\omega = 1/m$, we can write $\mathcal{P}^* l^m$ in the form

$$\mathcal{P}^* l^m f(z) = \mathcal{P}\chi\left[\left(\frac{z}{m}\right)^m I^{(k/m-1),(1)}_{(m),(m),m}\right] f(z) = \mathcal{P}\left[\frac{z}{m} I^{(k/m-1),(1)}_{(1),(1),m}\right]\chi f(z)$$
$$= c^* I^{(k/m-1),(\mu_k-k/m)}_{(\rho_k),(1),m}\left[m\cdot\frac{z}{m} I^{(k/m-1),(1)}_{(1),(1),m}\right]\chi f(z).$$

To simplify the above expression, it remains to apply *two of the basic operational rules of the generalized fractional integrals* of form (2.18), a kind of semi-group or product rules, namely (see (24) and (28) in [26])

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} z^\alpha f(z) = z^\alpha I^{(\gamma_k+\alpha/\lambda_k),(\delta_k+\alpha/\beta_k-\alpha/\lambda_k)}_{(\beta_k),(\lambda_k),m} f(z), \tag{4.14}$$

$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} = I^{(\gamma_k),(\delta_k+\sigma_k)}_{(\beta_k),(\lambda_k),m},$$
$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),(\lambda_k),m} I^{(\gamma_k),(\delta_k)}_{(\lambda_k),(\lambda_k),m} = I^{(\gamma_k),(\delta_k+\sigma_k)}_{(\beta_k),(\lambda_k),m}. \tag{4.15}$$

In this way, by (4.14) and (4.15), for $\mathcal{P}^* l^m f(z)$ we finally obtain

$$\mathcal{P}^* l^m f(z) = c^* I^{(k/m-1),(\mu_k-k/m)}_{(\rho_k),(1),m}\left[z I^{(k/m-1),(1)}_{(1),(1),m}\right]\chi f(z)$$
$$= c^* z I^{(k/m-1+1),(\mu_k-k/m+1/\rho_k-1)}_{(\rho_k),(1),m} I^{(k/m-1),(1)}_{(1),(1),m}\chi f(z)$$
$$= c^* z I^{(k/m-1),(\mu_k-k/m+1/\rho)}_{(\rho_k),(1),m}\chi f(z).$$

On the other hand, due to (4.15),

$$\mathfrak{L}\mathcal{P}^* f(z) = L_{(\rho_i),(\mu_i)}\mathcal{P}\chi f(z)$$
$$= z I^{(\mu_k-1),(1/\rho_k)}_{(\rho_k),(\rho_k),m}\left[c^* I^{(k/m-1),(\mu_k-k/m)}_{(\rho_k),(1),m}\right]\chi f(z)$$
$$= c^* z I^{(k/m-1),(\mu_k-k/m+1/\rho_k)}_{(\rho_k),(1),m}\chi f(z),$$

which is the same as the expression found for $\mathcal{P}^* l^m f(z)$. This completes the proof of the lemma. □

5. Solutions to Volterra-type integral equations of second kind

The Volterra integral equations of second kind

$$y(z) - \lambda\mathfrak{L}y(z) = y(z) - \lambda L_{(\rho_i),(\mu_i)} y(z) = f(z),$$

with operators $L_{(\rho_i),(\mu_i)}$ of form (3.3) or (3.4), are said to be *fractional multi-order integral equations of second kind*.

We suppose further that $\mu_k \geqslant 0,\ k = 1, \ldots, m$.

Theorem 1. *The unique solution $y(z) \in \mathcal{H}(\Omega)$ of the fractional multi-order integral equation of second kind, having the equivalent forms*

$$y(z) - \lambda L_{(\rho_i),(\mu_i)} y(z) = y(z) - \lambda z I^{(\mu_i - 1),(1/\rho_i)}_{(\rho_i),m} y(z)$$
$$= y(z) - \lambda z \int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\mu_i, 1/\rho_i) \\ (\mu_i - 1/\rho_i, 1/\rho_i) \end{matrix}\right] y(z\sigma)\, d\sigma = f(z), \tag{5.1}$$

$$y(z) - \lambda z \int_0^1 \dots \int_0^1 \left[\frac{(1-\sigma_i)^{1/\rho_i - 1}}{\Gamma(1/\rho_i)} \sigma_i^{\mu_i - 1}\right]$$
$$\times\, y\left[z(\sigma_1^{1/\rho_1} \dots \sigma_m^{1/\rho_m})\right] d\sigma_1 \dots d\sigma_m = f(z), \tag{5.2}$$

with $f \in \mathcal{H}(\Omega)$, is given by the series

$$y(z) = f(z) + \sum_{k=1}^{\infty} \lambda^k (L_{(\rho_i),(\mu_i)})^k f(z) := f(z) + \sum_{k=1}^{\infty} (\lambda z)^k H_k(z), \tag{5.3}$$

convergent for all $|z| < \infty$, where $H_k(z),\ k = 1, 2, \dots,$ stand for the integrals of H-functions

$$H_k(z) = \int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} \left(\mu_i + \frac{k-1}{\rho_i}, \frac{1}{\rho_i}\right)_{i=1}^m \\ \left(\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i}\right)_{i=1}^m \end{matrix}\right] f(z\sigma)\, d\sigma. \tag{5.4}$$

Proof. The homogeneous equation (5.1), (5.2) (i.e., $f \equiv 0$) has the only trivial solution $y \equiv 0$ and this yields the uniqueness of the solution in $\mathcal{H}(\Omega)$ in the nonhomogeneous case.

Consider first the simpler integral equation of second kind, involving the m-fold integration l^m, (4.7′), which is a R–L integral (2.8) of integer order $\delta = m > 1$:

$$\tilde{y}(z) - \lambda l^m \tilde{y}(z) = \tilde{f}(z), \quad \tilde{f} \in \mathcal{H}(\Omega). \tag{5.5}$$

This is a special case of (1.2) and its unique solution (1.3) has the form

$$\tilde{y}(z) = \tilde{f}(z) + \lambda \int_0^z (z-\zeta)^{m-1} E_{m,m}\left[\lambda (z-\zeta)^m\right] \tilde{f}(\zeta)\, d\zeta \in \mathcal{H}(\Omega).$$

Replacing the Mittag-Leffler function by its series definition (2.7), we obtain

$$\tilde{y}(z) = \tilde{f}(z) + \lambda \int_0^z (z-\zeta)^{m-1} \left[\sum_{k=0}^{\infty} \frac{\lambda^k (z-\zeta)^{mk}}{\Gamma(m(k+1))}\right] \tilde{f}(\zeta)\, d\zeta$$
$$= \tilde{f}(z) + \lambda \sum_{k=0}^{\infty} \lambda^k \left[\int_0^z \frac{(z-\zeta)^{m(k+1)-1}}{\Gamma(m(k+1))} \tilde{f}(\zeta)\, d\zeta\right]$$

$$= \tilde{f}(z) + \sum_{k=0}^{\infty} \lambda^{k+1} l^{m(k+1)} \tilde{f}(z)$$
$$= \tilde{f}(z) + \sum_{k=1}^{\infty} \lambda^{k} l^{mk} \tilde{f}(z) \in \mathcal{H}(\Omega). \tag{5.6}$$

In the above, we have changed the order of the integration and summation which is admissible, since the series representing the entire function $E_{m,m}[\lambda(z-\zeta)^m]$ is uniformly convergent and the terms $(z-\zeta)^{mk+m-1}\tilde{f}(\zeta) \in \mathcal{H}(\Omega)$ are integrable functions on $[0, z]$.

Let us apply the generalized Poisson transformation $\mathcal{P}^*$ ((4.6), (4.10)) to Eq. (5.5), denoting

$$\mathcal{P}^*\tilde{y}(z) := y(z) \in \mathcal{H}(\Omega), \qquad \mathcal{P}^*\tilde{f}(z) := f(z) \in \mathcal{H}(\Omega)$$
$$\big(\Rightarrow \quad \tilde{f}(z) = \mathcal{P}^{*-1} f(z) \in \mathcal{H}(\Omega)\big).$$

According to similarity relation (4.11), for $k = 1, 2, \ldots$ we have

$$\mathcal{P}^*(l^m)^k = (\mathcal{P}^* l^m)(l^m)^{k-1} = \mathfrak{L}\mathcal{P}^*(l^m)^{k-1}$$
$$= \mathfrak{L}(\mathcal{P}^* l^m)(l^m)^{k-2} = \mathfrak{L}^2\mathcal{P}^*(l^m)^{k-2} = \cdots = \mathfrak{L}^k\mathcal{P}^*.$$

Therefore,

$$\mathcal{P}^*\tilde{y}(z) - \lambda\mathcal{P}^* l^m \tilde{y}(z) = \mathcal{P}^*\tilde{f}(z)$$

turns into

$$y(z) - \lambda\mathfrak{L}y(z) = f(z);$$

that is, $\mathcal{P}^*$ transforms Eq. (5.5) into Eqs. (5.1), (5.2) and the solution (5.6) into the sought solution of the generalized fractional integral equation of second kind. Thus, we obtain

$$y(z) = \mathcal{P}^*\tilde{y}(z) = \mathcal{P}^*\left\{\tilde{f}(z) + \sum_{k=1}^{\infty} \lambda^k l^{mk} \tilde{f}(z)\right\}$$
$$= f(z) + \sum_{k=1}^{\infty} \lambda^k \big[\mathcal{P}^*(l^m)^k\big]\tilde{f}(z)$$
$$= f(z) + \sum_{k=1}^{\infty} \lambda^k \big[\mathfrak{L}^k\mathcal{P}^*\big]\tilde{f}(z)$$
$$= f(z) + \sum_{k=1}^{\infty} \lambda^k \mathfrak{L}^k f(z), \tag{5.7}$$

where we have put the generalized fractional integration $\mathcal{P}^*$ under the sign of the series $(\tilde{y}(z) - \tilde{f}(z)) \in \mathcal{H}(\Omega)$. This can be justified by the fact that its terms

$\lambda^k l^{mk}\tilde{f}(z)\in\mathcal{H}(\Omega)$ are integrable functions on $[0,z]$ and the resulting series is of absolutely convergent integrals $(k=1,2,\ldots)$:

$$\mathfrak{L}^k f(z)=z^k\int_0^1 H_{m,m}^{m,0}\left[\sigma\,\middle|\,\begin{matrix}\left(\mu_i+\frac{k-1}{\rho_i},\frac{1}{\rho_i}\right)_{i=1}^m\\ \left(\mu_i-\frac{1}{\rho_i},\frac{1}{\rho_i}\right)_{i=1}^m\end{matrix}\right]f(z\sigma)\,d\sigma:=z^k H_k(z).$$

The above follows from the more general integral representation of the powers of the generalized fractional integrals (from [11]) as the same type of generalized fractional integrals,

$$\left(z^{\delta_0}I_{(\beta_i),m}^{(\gamma_i),(\delta_i)}\right)^k f(z)=z^{k\delta_0}I_{(\beta_i),m}^{(\gamma_i),(k\delta_i)}f(z),$$

whose absolute convergence is a corollary of the conditions $\mu_i\geqslant 0,\ i=1,\ldots,m$, $f\in\mathcal{H}(\Omega)$ and the asymptotic behaviour of the kernel H-functions, as proved for (2.14) in the general case (cf. [11]).

Thus required solution (5.7) takes the form (5.3), (5.4). □

6. Solutions to fractional multi-order differential equations

The results of Theorem 1 allow us to find easily particular solutions of the *nonhomogeneous fractional multi-order differential equations* of the form

$$\begin{aligned}&\mathfrak{D}y(z)-\lambda y(z)\\&\quad=z^{-1}\prod_{i=1}^m\left(z^{1+(1-\mu_i)\rho_i}D_{z^{\rho_i}}^{1/\rho_i}\ z^{(\mu_i-1)\rho_i}\right)y(z)-\lambda y(z)=f(z).\end{aligned}\tag{6.1}$$

Namely, let us consider the integral equation of second kind (5.1),

$$\begin{aligned}&Y(z)-\lambda\mathfrak{L}Y(z)=F(z),\\&\text{with}\quad F(z):=\mathfrak{L}f(z)\in\mathcal{H}(\Omega)\quad\text{for } f(z)\in\mathcal{H}(\Omega).\end{aligned}\tag{6.2}$$

Applying the generalized fractional differentiation operator $\mathfrak{D}$ (3.3), (3.5), (3.6) to both sides of (6.2) and using $\mathfrak{D}\mathfrak{L}Y(z)=Y(z),\ Y\in\mathcal{H}(\Omega)$, we get

$$\mathfrak{D}Y(z)-\lambda Y(z)=f(z),$$

the fractional multi-order differential equation (6.1). Then, if $Y(z)$ is a solution of (6.2) with a right-hand side $F(z)=\mathfrak{L}f(z)$, it is also a solution of (6.1). According to (5.3),

$$\begin{aligned}Y(z)&=F(z)+\sum_{k=1}^\infty\lambda^k\mathfrak{L}^kF(z)=\mathfrak{L}f(z)+\sum_{k=1}^\infty\lambda^k\mathfrak{L}^{k+1}f(z)\\&=\sum_{k=0}^\infty\lambda^k\mathfrak{L}^{k+1}f(z)=z\sum_{k=0}^\infty(\lambda z)^kH_{k+1}(z),\end{aligned}$$

where for $\mathfrak{L}^{k+1}$ we use again the integral representation of the powers of the generalized fractional integral $\mathfrak{L}$ as in the proof of Theorem 1 but with $k \mapsto k+1$.

Thus, the following theorem holds.

Theorem 2. *A particular solution* $Y(z) \in \mathcal{H}(\Omega)$ *to the fractional multi-order nonhomogeneous differential equation* (6.1),

$$\mathfrak{D}Y(z) - \lambda Y(z) = f(z), \quad f \in \mathcal{H}(\Omega),$$

is given by the series

$$Y(z) = z \sum_{k=0}^{\infty} (\lambda z)^k H_{k+1}(z), \tag{6.3}$$

convergent for $|z| < \infty$, *with* $H_{k+1}(z)$, $k = 0, 1, 2, \ldots$, *as in* (5.4).

Let us mention that finding the *form of the general solution* of the nonhomogeneous fractional differential equations (6.1) (i.e., satisfying arbitrary initial conditions) leads to the yet *open problem* for a *fundamental system of solutions of equation* $\mathfrak{D}y(z) = \lambda y(z)$. It may be treated by applying the generalized Poisson transformation $\mathcal{P}^*$ to the system of the other (linearly independent) solutions of the simpler equation ($\widetilde{\mathbf{E}}$), given by the *generalized* (m*-order*) *sine functions*:

$$\sin_{m,m-i+1}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{mk+i-1}}{(mk+i-1)!} \in \mathcal{H}\big(\{|z| < \infty\}\big),$$
$$i = 2, \ldots, m; \tag{6.4}$$

see [11, p. 338], [19, Vol. 3].

7. Examples

A. First we consider fractional multi-order integral and differential equations in their general form, taking only some specific analytic functions as the right-hand sides $f(z)$.

Example A.1. Let $f(z) = z^p$, $p \geqslant 0$.

Then, the expressions $H_k(z)$ can be easily evaluated, similarly to these in (2.2), by using an integral formula for the $H^{m,0}_{m,m}$-functions (see [11, p. 348, (E.21)]) and the properties of the H-functions:

$$H_k(z) = z^p \int_0^1 H^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} \left(\mu_i + \frac{k-1}{\rho_i}, \frac{1}{\rho_i}\right) \\ \left(\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i}\right) \end{matrix}\right] \sigma^p \, d\sigma$$

$$
\begin{aligned}
&= z^p \int_0^1 H_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} \left(\mu_i + \frac{k+p-1}{\rho_i}, \frac{1}{\rho_i}\right) \\ \left(\mu_i + \frac{p-1}{\rho_i}, \frac{1}{\rho_i}\right) \end{matrix}\right] d\sigma \\
&= z^p \prod_{i=1}^m \frac{\Gamma\left(\mu_i + \frac{p}{\rho_i}\right)}{\Gamma\left(\mu_i + \frac{k+p}{\rho_i}\right)} := z^p c_{p,k},
\end{aligned}
$$

$$\text{resp.} \quad H_{k+1}(z) = z^p c_{p,k+1}. \tag{7.1}$$

Therefore, the solutions (5.3) and (6.3) become, respectively,

$$
\begin{aligned}
y(z) &= z^p \left\{ 1 + \sum_{k=1}^{\infty} (\lambda z)^k c_{p,k} \right\} \\
&= z^p \left\{ 1 + \prod_{i=1}^m \Gamma\left(\mu_i + \frac{p}{\rho_i}\right) \sum_{k=1}^{\infty} \frac{(\lambda z)^k}{k!} \frac{\Gamma(1+k)}{\prod_{i=1}^m \Gamma\left(\mu_i + \frac{p}{\rho_i} + k\frac{1}{\rho_i}\right)} \right\} \\
&= C_p \cdot z^p \, {}_1\Psi_m \left[\begin{matrix} (1,1) \\ \left(\mu_i + \frac{p}{\rho_i}, \frac{1}{\rho_i}\right) \end{matrix} \,\middle|\, \lambda z \right]
\end{aligned}
$$

$$\text{with} \quad C_p = \prod_{i=1}^m \Gamma\left(\mu_i + \frac{p}{\rho_i}\right), \tag{7.2}$$

and

$$
\begin{aligned}
Y(z) &= C_p z^{p+1} \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{k!} \frac{\Gamma(1+k)}{\prod_{i=1}^m \Gamma\left(\mu_i + \frac{p+1}{\rho_i} + k\frac{1}{\rho_i}\right)} \\
&= C_p z^{p+1} \, {}_1\Psi_m \left[\begin{matrix} (1,1) \\ \left(\mu_i + \frac{p+1}{\rho_i}, \frac{1}{\rho_i}\right) \end{matrix} \,\middle|\, \lambda z \right].
\end{aligned} \tag{7.3}
$$

Example A.2. Let the RHS of (6.1),

$$
\begin{aligned}
& f(z) = H_{\gamma,\delta}^{\alpha,\beta}\left[\omega z^r \,\middle|\, \begin{matrix} (a_j, A_j)_1^\gamma \\ (b_j, B_j)_1^\delta \end{matrix}\right] \in \mathcal{H}\big(\{|z| < R\}\big), \\
& \quad \omega \neq 0, \quad r > 1,
\end{aligned}
$$

be an arbitrary H-function. Then, practically all the particular right-hand sides of the generalized integral and differential equations that could appear as $f(z)$ are encompassed.

To evaluate $H_k(z)$, we observe that for the parameters chosen, $H_{m,m}^{m,0}(\sigma) = 0$ for $|\sigma| > 1$ and consequently, the integral from 0 to 1 in $H_k(z)$ can be replaced by an integral from 0 to infinity, of a product of two H-functions. Applying the known formulas for such integrals (see [11, p. 348, (E.21′); 18]), we obtain

$$
H_k(z) = \int_0^{\infty} H_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} \left(\mu_i + \frac{k-1}{\rho_i}, \frac{1}{\rho_i}\right)_{i=1}^m \\ \left(\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i}\right)_{i=1}^m \end{matrix}\right] H_{\gamma,\delta}^{\alpha,\beta}\left[\omega z^r \sigma^r \,\middle|\, \begin{matrix} (a_j, A_j) \\ (b_j, B_j) \end{matrix}\right] d\sigma
$$

$$= H^{\alpha,\beta+m}_{\gamma+m,\delta+m}\left[\omega z^r \,\middle|\, \begin{matrix} (a_j, A_j)_1^\beta, \left(1-\mu_i, \frac{r}{\rho_i}\right)_1^m, (a_j, A_j)_{\beta+1}^\gamma \\ (b_j, B_j)_1^\alpha, \left(1-\mu_i-\frac{k}{\rho_i}, \frac{1}{\rho_i}\right)_1^m, (b_j, B_j)_{\alpha+1}^\delta \end{matrix}\right]; \tag{7.4}$$

therefore the solutions (5.3) and (6.3) of Eqs. (5.1), (6.1) appear as convergent series of H-functions, analytic in the same disks of $\mathbb{C}$ as the H-functions for $f(z)$,

$$y(z) = H^{\alpha,\beta}_{\gamma,\delta}\left[\omega z^r \,\middle|\, \begin{matrix} (a_j, A_j)_1^\gamma \\ (b_j, B_j)_1^\delta \end{matrix}\right] + \sum_{k=1}^{\infty}(\lambda z)^k \times H^{\alpha,\beta+m}_{\gamma+m,\delta+m}\left[\omega z^r \,\middle|\, \begin{matrix} (a_j, A_j)_1^\beta, \left(1-\mu_i, \frac{r}{\rho_i}\right)_1^m, (a_j, A_j)_{\beta+1}^\gamma \\ (b_j, B_j)_1^\alpha, \left(1-\mu_i-\frac{k}{\rho_i}, \frac{1}{\rho_i}\right)_1^m, (b_j, B_j)_{\alpha+1}^\delta \end{matrix}\right], \tag{7.5}$$

respectively,

$$Y(z) = z\sum_{k=0}^{\infty}(\lambda z)^k \times H^{\alpha,\beta+m}_{\gamma+m,\delta+m}\left[\omega z^r \,\middle|\, \begin{matrix} (a_j, A_j)_1^\beta, \left(1-\mu_i, \frac{r}{\rho_i}\right)_1^m, (a_j, A_j)_{\beta+1}^\gamma \\ (b_j, B_j)_1^\alpha, \left(1-\mu_i-\frac{k+1}{\rho_i}, \frac{1}{\rho_i}\right)_1^m, (b_j, B_j)_{\alpha+1}^\delta \end{matrix}\right]. \tag{7.6}$$

Next, we can specify the H-function to be an *multi-index Mittag-Leffler function with the same parameters* as in the operators $L_{(\rho_i),(\mu_i)}$, $D_{(\rho_i),(\mu_i)}$.

Example A.3. Take $f(z) = E_{(1/\rho_i),(\mu_i)}(z^r) \in \mathcal{H}(\{|z| < \infty\})$, $r > 0$. By (7.4), (2.7′) and the other properties of the H-functions, it follows

$$\begin{aligned} H_k(z) &= H^{1,m+1}_{m+1,2m+1}\left[-z^r \,\middle|\, \begin{matrix} (0,1), \left(1-\mu_i, \frac{1}{\rho_i}\right)_1^m \\ (0,1), \left(1-\left(\mu_i+\frac{k}{\rho_i}\right), \frac{1}{\rho_i}\right)_1^m, \left(1-\mu_i, \frac{1}{\rho_i}\right)_1^m \end{matrix}\right] \\ &= H^{1,1}_{1,m+1}\left[-z^r \,\middle|\, \begin{matrix} (0,1) \\ (0,1), \left(1-\left(\mu_i+\frac{k}{\rho_i}\right), \frac{1}{\rho_i}\right)_1^m \end{matrix}\right] \\ &= E_{(1/\rho_i),(\mu_i+k/\rho_i)}(z^r); \end{aligned} \tag{7.7}$$

therefore the solutions (5.3), (6.3) are expressed as series of multi-index Mittag-Leffler functions. For example, a particular solution of the nonhomogeneous fractional multi-order differential equation

$$D_{(\rho_i),(\mu_i)}Y(z) - \lambda Y(z) = E_{(1/\rho_i),(\mu_i)}(z^r),$$

is

$$Y(z) = z\sum_{k=0}^{\infty}(\lambda z)^k E_{(1/\rho_i),(\mu_i+k/\rho_i)}(z^r) \in \mathcal{H}\big(\{|z| < \infty\}\big). \tag{7.8}$$

B. Now let us consider the special case of the fractional multi-order operators $\mathfrak{D}, \mathfrak{L}$, when the multi-order of differentiation or integration $(1/\rho_1, 1/\rho_2, \ldots, 1/\rho_m)$ consists of integers, and especially is $(1, 1, \ldots, 1)$. In this case we obtain the *hyper-Bessel differential and integral operators*. Namely, by setting $\mu_i = \gamma_i + 1$, $\rho_i = 1$, $i = 1, \ldots, m$, we obtain the hyper-Bessel differential operator B as in (3.7), with $\beta = 1$, and its corresponding hyper-Bessel integral operator (in the denotations (2.14)),

$$\begin{aligned} Lf(z) &= z I_{(1),m}^{(\gamma_1,\ldots,\gamma_m),(1,\ldots,1)} f(z) \\ &= z \int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_i + 1)_1^m \\ (\gamma_i)_1^m \end{matrix}\right] f(z\sigma)\, d\sigma. \end{aligned} \tag{7.9}$$

The solutions (5.3), (6.3) of the respective hyper-Bessel integral equations of second kind and of the nonhomogeneous differential equations of order $m > 1$ contain Meijer's G-functions instead of the H-functions, and the solution of the homogeneous equation $By(z) = \lambda y(z)$ is given in terms of the hyper-Bessel function (2.4) (see, e.g., [11, Ch. 3; 13]).

In particular, for special choice of $f(z) = z^p$ we obtain, for the solutions of the hyper-Bessel integral and differential equations,

$$\begin{aligned} y(z) &= z^p\, {}_1F_m\big(1; (\gamma_i + p + 1)_1^m; \lambda z\big), \\ Y(z) &= z^{p+1}\, {}_1F_m\big(1; (\gamma_i + p + 2)_1^m; \lambda z\big), \end{aligned} \tag{7.10}$$

and in the case of $f(z) = G_{\gamma,\delta}^{\alpha,\beta}(z)$, the solutions are simpler *series of Meijer's G-functions*. Especially, an analogue of Example A.3 can be derived when $f(z)$ is a hyper-Bessel function with the same parameters as for the operators B, L and then the solutions appear as series of the same kind of hyper-Bessel functions.

C. Consider the special case of hyper-Bessel operators related to the Bessel function $J_\nu(z)$ and to the classical second order Bessel differential equation; namely take $m = 2$; $\gamma_i = \pm\nu/2$, $i = 1, 2$; $\beta = 2$ (adjusting to the case $\beta = 1$ comes with the simpler substitution, as in (4.8), $\chi : z \mapsto 2\sqrt{z}$). The *Bessel differential operator* is, in the form (3.7),

$$B_\nu = z^{-2}\left(z^{\nu+1}\frac{d}{dz} z^{-\nu} z^{-\nu+1}\frac{d}{dz} z^{\nu}\right) = \frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - \frac{\nu^2}{z^2},$$

and the hyper-Bessel integral equation of second kind, $y(z) - \lambda L_\nu y(z) = f(z)$, contains the *hypergeometric* (*with the Gauss function as a kernel*) *integral operator*

$$\begin{aligned} L_\nu y(z) = \left(\frac{z}{2}\right)^2 \int_0^1 & \sigma^{-\nu/2}(1-\sigma) \\ & \times {}_2F_1(1-\nu, 1; 2; 1-\sigma)\, y\big(z\sqrt{\sigma}\big)\, d\sigma. \end{aligned} \tag{7.11}$$

For special choices of $f(z)$ as in Example A.1, one obtains the *Lommel* and the *Struve functions* as solutions of (7.11), namely

$$\text{(i) } f(z) = z^{\mu+1} \quad \Rightarrow$$

$$y(z) = z^{\mu+1}\, {}_1F_2\left(1; \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; -\frac{z^2}{4}\right) = (\mu+\nu+1)(\mu-\nu+1)s_{\mu,\nu}(z),$$

$$\text{(ii) } f(z) = z^{\nu+1} \quad \Rightarrow$$

$$y(z) = z^{\nu+1}\, {}_1F_2\left(1; \nu+\frac{3}{2}, \frac{3}{2}; -\frac{z^2}{4}\right) = (2\nu+1)\, s_{\nu,\nu}(z) = \sqrt{\pi}\,\Gamma(\nu+1/2)2^{\nu-1}(2\nu+1)H_\nu(z).$$

D. For $m = 1$, the problems considered here are related to the *classical Mittag-Leffler function* as in (1.2), (1.3).

E. Take in general, $m = 2$; the case is related to *Dzrbashjan's Mittag-Leffler function* from [29]. Special cases of it are related to *Bessel–Maitland functions*, etc.

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